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# Indecomposable knots and concordance

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## *Introduction*

R. C. Kirby and W. B. R. Lickorish have proved (cf. (4)) that any classical knot is concordant to an indecomposable knot. In the present note we show that this statement is also true for higher dimensional knots: more precisely, for any higher-dimensional knot $K$ there exist infinitely many non-isotopic indecomposable simple knots which are concordant to $K$. This, together with the result of Kirby and Lickorish, gives a complete solution of problem 13 of (1).

## 1. *Simple knots and isometric structures*

Kervaire has proved that the concordance group of even dimensional knots is trivial (cf. (2)), so we only need to consider odd-dimensional knots $K^{2q-1} \subset S^{2q+1}$, $q > 1$. Such a knot $K^{2q-1}$ is said to be *simple* if $\pi_i(S^{2q+1}\backslash K^{2q-1}) \cong \pi_i(S')$ for $i < q$. Levine has proved that every higher-dimensional $(2q-1)$-knot is concordant to a simple knot (cf. (5)).

An *isometric structure* is a triple $(L, \langle , \rangle, z)$ where $L$ is a free $\mathbb{Z}$-module of finite rank, $\langle , \rangle : L \times L \rightarrow \mathbb{Z}$ is a $\mathbb{Z}$-bilinear, $\epsilon$-symmetric (where $\epsilon = \pm 1$) non-singular form (i.e. the adjoint of $\langle , \rangle$ is an isomorphism), $z: L \rightarrow L$ is a $\mathbb{Z}$-linear endomorphism such that $\langle za, b\rangle = \langle a, (1-z)\, b\rangle$ for all $a, b \in L$.

Two isometric structures are *isomorphic* if there exists a $\mathbb{Z}$-linear isomorphism which is an isomorphism between the forms and also commutes with the endomorphisms.

An isometric structure $(L, \langle , \rangle, z)$ is *metabolic* if $L$ contains a sub $\mathbb{Z}$-module $M$ which is stable by $z$, such that $\operatorname{rank}(L) = 2 \operatorname{rank}(M)$ and that $\langle a, b\rangle = 0$ for all $a, b$ in $M$. Two isometric structures $L_1$ and $L_2$ are *Witt-equivalent* if there exist metabolic isometric structures $N_1, N_2$ such that $L_1 \perp N_1 \cong L_2 \perp N_2$ (where $\perp$ denotes orthogonal sum).

To any odd-dimensional knot $K^{2q-1}$ we can associate an isometric structure with $\epsilon = (-1)^q$ (cf. (5) and (3)). Levine has proved that two simple $(2q-1)$-knots, $q > 1$, are concordant if and only if the corresponding isometric structures are Witt-equivalent (cf. (5)). The isometric structure of a connected sum is the orthogonal sum of isometric structures.

An isometric structure $(L, \langle , \rangle, z)$ is said to be decomposable if $L \otimes \mathbb{Z}[1/a] \cong (L_1 \otimes \mathbb{Z}[1/a]) \perp (L_2 \otimes \mathbb{Z}[1/a])$ where $L_1, L_2$ are non-trivial isometric structures and $a = \det(z)$. Otherwise, we say that $(L, \langle , \rangle, z)$ is *indecomposable*. This definition is

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motivated by the fact that if two simple knots are isotopic, then the corresponding isometric structures are $S$-equivalent (cf. (6)) and therefore $\mathbb{Z}[1/a]$-isomorphic (cf. (9)).

Therefore it suffices to prove the following:

THEOREM. *Every isometric structure* $(L, \langle, \rangle, z)$ *is Witt-equivalent to an indecomposable isometric structure.*

*Addendum.* The Witt-equivalence class of $(L, \langle, \rangle, z)$ contains infinitely many indecomposable isometric structures which all have distinct characteristic polynomials.

Indeed, every $(-1)^q$-symmetric isometric structure can be realized by a simple $(2q-1)$-knot (cf. (3)) and the characteristic polynomial of the isometric structure is related to the Alexander polynomial of the knot (cf. (8), p. 14) which is an invariant of the isotopy class of the knot.

## 2. *Proof of the Theorem*

Let $(L, \langle, \rangle, z)$ be an isometric structure. If $f \in \mathbb{Z}[x]$, we shall denote $f^*(x) = f(1-x)$. By (8), Proposition 1·8, we may assume that the minimal polynomial $\phi$ of $z$ can be written $\phi = \phi_1 \dots \phi_n$, $\phi_i^* = \phi_i$, $\phi_i$ irreducible and $\phi_i \neq \phi_j$ if $i \neq j$. Let $p$ be a prime such that $\mathbb{Z}_{(p)}[x]/(\phi_i)$ is Dedekind $(i = 1, \dots, n)$ that $p$ does not divide $\phi(0)$ and that $p$ does not divide any of the resultants $\operatorname{Res}(\phi_i, \phi_j)$. Let $\psi_1, \dots, \psi_k$ be polynomials with integral coefficients such that $\psi_1 = \phi$, that $\psi_{i+1}$ divides $\psi_i$ for all $i = 1, \dots, k-1$ and that the product $\psi_1 \dots \psi_k$ is the characteristic polynomial of $z$. Let $F_i \in \mathbb{Z}[x]$ be monic, irreducible polynomials such that $F_i \neq F_i^*$, that $F_i \neq F_j$ if $i \neq j$ and $F_i \equiv \psi_i \bmod p^2$ for all $i = 1, \dots, k$ (apply the chinese remainder theorem).

Let $H_i = \operatorname{Hyp}(\mathbb{Z}[x]/(F_i))$ be the hyperbolic isometric structure associated to the polynomial $F_i$, i.e. the isometric structure

$$\left(\mathbb{Z}^{2n_i}, \begin{pmatrix} 0 & I \\ \epsilon I & 0 \end{pmatrix}, \begin{pmatrix} M_i & 0 \\ 0 & I - M_i^t \end{pmatrix}\right),$$

where $n_i = \deg(F_i)$, $I$ is the identity matrix, $M_i$ is the companion matrix to $F_i$ and $M_i^t$ is the transpose of $M_i$.

Let $H' = H_1 \perp \dots \perp H_k$.

The first step will be to construct an indecomposable, metabolic isometric structure $M$ such that $H'_{(p)}$ is an orthogonal summand of $M_{(p)}$ (notice that if $k = 1$, i.e. if the characteristic polynomial of $z$ has no repeated factor, then we can take $M = H'$).

Let $q \neq p$ be a prime which does not divide $\phi(0)$, $F_i(0)$, $F_i^*(0)$ for all $i = 1, \dots, k$, and does not divide any of the resultants $\operatorname{Res}(F_i F_i^*, F_j F_j^*)$, $\operatorname{Res}(\phi_i, F_j F_j^*)$. Let $F \in \mathbb{Z}[x]$ be a monic irreducible polynomial such that $F = F^*$, that $F(0)$, $F^*(0)$ are not divisible by $p$ and that

$$F \equiv \prod_{i=1}^{k} F_i F_i^* \bmod q^2.$$

Let $H'' = \operatorname{Hyp}(\mathbb{Z}[x]/(F))$ be the hyperbolic isometric structure associated to $F$. Set $H = H' \perp H''$. Notice that $(qH)^\# = (1/q)H$ (if $K$ is a lattice in $(V^q, [,])$, then we denote $K^\# = \{x \in V \text{ such that } [x, K] \subseteq \mathbb{Z}\}$).

Let $T = (1/q)H/qH$, together with the induced torsion isometric structure (cf. (8), Section 2).

We have $T = T_{H'} \perp T_{H''}$, where $T_{H'} = (1/q)H'/qH'$, $T_{H''} = (1/q)H''/qH''$. Let $\pi: (1/q)H \to T$ be the projection. Multiplication by $q$ gives the isomorphism:

$$T_{H'} \cong \perp_{i=1,\dots,k} \mathrm{Hyp}(\mathbb{Z}/q^2[x]/(F_i)) = \perp_{i=1,\dots,k} (T_{F_i F_i^*}, S_i, z_i).$$

Notice that $T_{F_i F_i^*}$ is a free $\mathbb{Z}/q^2$-module.

Similarly,

$$T_{H''} \cong \mathrm{Hyp}((\mathbb{Z}/q^2)[x]/(F)) \cong \perp_{i=1,\dots,k} \mathrm{Hyp}((\mathbb{Z}/q^2)[x]/(F_i F_i^*)).$$

This decomposition follows from an analogue of (8), Theorem 3·2, because $q$ does not divide any of the resultants $\mathrm{Res}(F_i F_i^*, F_j F_j^*)$.

We shall need the following

*Claim* 2·1. Let $R$ be a commutative ring and $F = R^m$. Let $(F, S, w)$ be an isometric structure. Then

$$\left(F \oplus F, \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}, \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}\right)$$

is isomorphic to

$$\left(F \oplus F, \begin{pmatrix} 0 & I \\ \epsilon I & 0 \end{pmatrix}, \begin{pmatrix} w & 0 \\ 0 & 1-w^t \end{pmatrix}\right).$$

Indeed, an isomorphism is given by

$$\begin{pmatrix} I & 0 \\ wS & I \end{pmatrix}\begin{pmatrix} S^{-1} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} = \begin{pmatrix} S^{-1} & S^{-1} \\ w & 1-w \end{pmatrix}.$$

We shall apply this claim with $R = \mathbb{Z}/q^2$, $F = T_{F_i F_i^*}$, $S = S_i$, $w = z_i$. We obtain:

$$T_{H''} \cong \perp_{i=1,\dots,k} \{(T_{F_i F_i^*}, S_i, z_i) \perp (T_{F_i F_i^*}, -S_i, z_i)\}.$$

Accordingly, we have a decomposition of $T$

$$T = \perp_{i=1,\dots,k} (T_i \perp T_i \perp (-T_i)) \quad \text{with} \quad T_i = (T_{F_i F_i^*}, S_i, z_i).$$

Let

$$X_i = \{(x, qy, x) \in T_i \perp T_i \perp (-T_i)\} \quad \text{and} \quad X = \bigoplus_{i=1}^{k} X_i.$$

Set

$$M = \pi^{-1}(X) \subset \frac{1}{q} H.$$

As $X$ is a metabolizer in $T$, we have $M^\# = M$. Clearly $M$ is metabolic (cf. (8), proposition 1·3). Moreover, $M_{(q)}$ is indecomposable (where $M_{(q)} = M \otimes \mathbb{Z}_{(q)}$). Indeed, assume that $M_{(q)} = M_1 \perp M_2$. Since the decomposition is orthogonal, we may assume that $FF^*$ divides the characteristic polynomial of $M_2$. Then $M_1 \subseteq \mathrm{Ann}_G(M_{(q)})$, where $G = (F_1 \dots F_k)(F_1 \dots F_k)^*$. We have $M_1 \subset (1/q)H'_{(q)}$, but $X \cap \pi((1/q)H'_{(q)}) = \{0\}$, therefore $M_1 \subset qH'_{(q)}$. This contradicts the assumption that $M_1^\# = M_1$. Finally, it is clear that $M_{(r)} = H_{(r)}$ if $r \neq q$ because $M \subset (1/q)H$.

Let $T' = (1/p)(L \perp M)/p(L \perp M) = T_L \perp T_M$, and let $\rho: (1/p)(L \perp M) \to T'$ be the projection. Let $L_i = \mathrm{Ann}_{\phi_i}(L_{(p)})$. We have

$$L_{(p)} = \perp_{i=1,\dots,n} L_i,$$

because $p$ does not divide $\mathrm{Res}(\phi_i, \phi_j)$ if $i \neq j$. Let $\Lambda_i = \mathbb{Z}_{(p)}[X]/(\phi_i)$. Then $L_i$ is a torsion free $\Lambda_i$-module of finite rank. We have assumed that $\Lambda_i$ is Dedekind, therefore $L_i$ is projective. Moreover $\Lambda_i$ is semi-local, so $L_i$ is a free $\Lambda_i$-module of rank, say, $m_i$ (cf. (7), p. 24). We have:

$$T_L \cong \coprod_{i=1,\ldots,n} L_i/p^2L_i \cong \coprod_{i=1,\ldots,n} (T_{\phi_i}, S_i', z_i'),$$

where $T_{\phi_i}$ is a free $\mathbb{Z}/p^2[X]/(\phi_i)$-module of rank $m_i$.

On the other hand, we have:

$$\begin{aligned} T_M = \frac{1}{p} M_{(p)}/pM_{(p)} &\cong (\coprod_{j=1,\ldots,k} H_j/p^2H_j) \perp T'' \\ &\cong \coprod_{i=1,\ldots,n} [\mathrm{Hyp}\,(\mathbb{Z}/p^2[X]/(\phi_i))]^{m_i} \perp T'' \\ &\cong \coprod_{i=1,\ldots,n} [\mathrm{Hyp}\,(\mathbb{Z}/p^2[X]/(\phi_i))^{m_i}] \perp T'', \end{aligned}$$

where $T'' = H''/p^2H''$.

Applying Claim 2·1, we have

$$\mathrm{Hyp}\,(\mathbb{Z}/p^2[X]/(\phi_i))^{m_i} \cong (T_{\phi_i}, S_i', z_i') \perp (T_{\phi_i}, -S_i', z_i').$$

This induces a splitting of $T'$:

$$T' \cong \coprod_{i=1,\ldots,n} (T_i \perp T_i' \perp (-T_i')) \perp T''$$

let

$$Y_i = \{(x, py, x) \in T_i' \perp T_i' \perp (-T_i')\}$$

and

$$Y = \left[\bigoplus_{i=1}^{n} Y_i\right] \oplus pT''.$$

Set

$$N = \rho^{-1}(Y) \subset \frac{1}{p}(L \perp M).$$

As $Y$ is a metabolizer in $T'$, we have $N^{\#} = N$. It is clear that $N$ is Witt-equivalent to $L \perp M$, because they are Witt-equivalent over the rationals (cf. (8), 1·3 and 1·6), so $N$ and $L$ are Witt-equivalent.

Let us check that $N$ is indecomposable. Let $a = \det(z)\, F_i F_i^*(0) \ldots F_k F_k^*(0)\, FF^*(0)$. Assume that $N \otimes \mathbb{Z}[1/a] = (N' \otimes \mathbb{Z}[1/a]) \perp (N'' \otimes \mathbb{Z}[1/a])$. Suppose that $F$ divides the characteristic polynomial of $N''$. Then $F_1 \ldots F_k$ also divides the characteristic polynomial of $N''$. Indeed, if $F_i$ divides the characteristic polynomial of $N'$, then

$$K = \mathrm{Ann}_{F_i F_i^*}(N \otimes \mathbb{Z}[1/a]) \subset N' \otimes \mathbb{Z}[1/a].$$

As $q$ is prime to $a$, to $\mathrm{Res}\,(\phi_j, F_i F_i^*)$ and to $\mathrm{Res}\,(F_i F_i^*, F_j F_j^*)$ for all $j$, $K_{(q)}$ is an orthogonal summand of $N'_{(q)}$. But $N_{(q)} = L_{(q)} \perp M_{(q)}$ and $K_{(q)}$ is contained in $M_{(q)}$; therefore this gives a splitting of $M_{(q)}$ which is impossible. So we have

$$N' \otimes \mathbb{Z}\left[\frac{1}{a}\right] \subset \mathrm{Ann}_\phi\left(N \otimes \mathbb{Z}\left[\frac{1}{a}\right]\right) \subset \frac{1}{p}\left(L \otimes \mathbb{Z}\left[\frac{1}{a}\right]\right).$$

But

$$Y \cap \rho\left(\frac{1}{p}L\right) = \{0\},$$

therefore

$$N' \otimes \mathbb{Z}\left[\frac{1}{a}\right] \subset p\left(L \otimes \mathbb{Z}\left[\frac{1}{a}\right]\right),$$

so we cannot have $(N')^{\#} = N'$.

*Proof of the addendum.* The characteristic polynomial of the isometric structure $N$ is $\psi_1 \ldots \psi_k F_1 \ldots F_k F_1^* \ldots F_k^* F F^*$. It is easy to see that we have infinitely many possibilities for, say, $F_1$.

## 3. *Example*

An explicit illustration of the method is given in the following example:

Consider the direct sum of the isometric structures

$$L_1 = \left(\mathbb{Z} \oplus \mathbb{Z}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right)$$

and

$$L_2 = \left(\mathbb{Z} \oplus \mathbb{Z}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\right)$$

with minimal polynomials $\phi_1(X) = X^2 - X + 1$ and $\phi_2(X) = X^2 - X + 2$, respectively.

To find an indecomposable isometric structure by our method, we choose a polynomial $F \equiv \phi_1 \bmod 4$ and also $F \equiv \phi_2 \bmod 9$. One choice is $F(X) = X^2 + 35X + 29$, and we let $M = \mathbb{Z}[X]/(F(X))$. We construct the desired indecomposable as a lattice inside the rational vector space spanned by

$$L = \operatorname{Hyp}(M) \perp L_1 \perp L_2$$

as follows:

The dual lattice to $6L$ is $\frac{1}{6}L$ with the quotient, $T$, isomorphic to $L/36L$ with the isometric structure obtained by reduction modulo 36. Now $T$ splits orthogonally into its $p$-primary components $T_2$ and $T_3$ inside which we choose metabolic subgroups in the following manner: Since the endomorphisms of $M$ and $L_1$ have the same mod 4 reduction it follows from Claim 2·1 that

$$T_2 \cong \operatorname{Hyp}(\bar{M}) \oplus \bar{L}_1 \oplus \bar{L}_2 \cong \bar{L}_1 \oplus -\bar{L}_1 \oplus \bar{L}_1 \oplus \bar{L}_2$$

and we may choose the metabolic subgroup

$$H_2 = \{(2x, y, y, 2z) \in \bar{L}_1 \oplus -\bar{L}_1 \oplus \bar{L}_1 \oplus L_2\}$$

inside $T_2$. Similarly for $H_3 \subset T_3$.

The desired indecomposable structure is $N = \pi^{-1}(H_2 \oplus H_3)$ where

$$\pi : \tfrac{1}{6}L/6L \to T = T_2 \oplus T_3$$

is the projection.

First we give the isometric structure for $\hat{L} = \pi^{-1}(H_2 \oplus 3T_3)$ which reveals how $L_1$ is 'tied' to $\operatorname{Hyp}(M)$: $\hat{L} = (\mathbb{Z}^8, \hat{S}, \hat{z})$, where

$$\hat{S} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\hat{z} = \begin{pmatrix} -28 & -129 & 128 & 200 & -32 & -50 & 0 & 0 \\ 1 & 29 & 0 & -56 & 0 & 14 & 0 & 0 \\ -14 & -50 & 64 & 71 & -16 & -18 & 0 & 0 \\ 0 & 32 & 1 & -63 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Finally the isometric structure for $N = (\mathbb{Z}^8, S, z)$:

$$S = \begin{pmatrix} 0 & 13 & -20 & -12 & 0 & -9 & 0 & 12 \\ -13 & 0 & 0 & 8 & 6 & 6 & 0 & -8 \\ 20 & 0 & 0 & -12 & -9 & -9 & 0 & 12 \\ 12 & -8 & 12 & 0 & -3 & 0 & 4 & 0 \\ 0 & -6 & 9 & 3 & 0 & 4 & 0 & 0 \\ 9 & -6 & 9 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & 9 \\ -12 & 8 & -12 & 0 & 0 & 0 & -9 & 0 \end{pmatrix},$$

$$z = \begin{pmatrix} -42 & 537 & 592 & -1515 & 432 & 612 & -256 & -676 \\ 9 & -135 & -141 & 365 & -96 & -162 & 56 & 164 \\ -14 & 246 & 241 & -633 & 144 & 306 & -84 & -284 \\ 0 & 32 & 21 & -62 & 0 & 48 & 0 & -28 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Partial credit is awarded to SPEAKEASY for help in the matrix manipulations.

Notice that $(\mathbb{Z}^8, S, Z) \otimes \mathbb{Z}[\frac{1}{2}]$ is decomposable. As $\det(z) = 3770$, this implies that $N = (\mathbb{Z}^8, S, z)$ is decomposable in the sense of our definition of Section 1. However, $N$ is not $S$-equivalent to an orthogonal sum of non-trivial isometric structures (the $S$-equivalence of isometric structures being defined as the $S$-equivalence of the associated Seifert matrices, cf. (6), (10)).

Indeed, $\det(Z) = 3770$ is square-free. Trotter has proved that two isometric structures with square-free determinant are $S$-equivalent if and only if they are isomorphic over $\mathbb{Z}_{(p)}$ for all primes $p$ (cf. (10), Corollary 4·7$a$). Now it is easy to check that $N$ does not decompose in the same way over $\mathbb{Z}_{(2)}$ and $\mathbb{Z}_{(3)}$.

Therefore the simple $(4q+1)$-knot, $q \geqslant 1$, associated to $N$ is indecomposable.

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